Math 432 Lec 04 Recurrence Relations

Let a_n count an object formed from n elements. Like in induction, we sometimes can build a relation $a_n = f(a_{n-1}, a_{n-2}, \dots, a_{n-k})$ between a_n and $a_{n-1}, \dots, a_{n-k}$. This is called a recurrence relation.

(1) Let $W(n)$ be the number of simple words formed from n different letters. We build a relation between $W(n)$ and $W(n-1)$.

Claim: $W(n) = 1 + n \cdot W(n-1)$ and $W(0) = 1$. The condition $W(0) = 1$ is because of the empty word. In general, to form a word of from n letters, we choose one letter to go first (in n ways), and make a word from the remaining $n-1$ letters (in $W(n-1)$ ways) to follow it; but we have missed out one word, namely the empty word, so we need to add 1.

(2) (The Hanoi Tower Problem) n rings on a peg must be moved to another peg, with a third peg available as workspace. No ring can ever be placed on a smaller ring. Let a_n be the minimum number of steps required to move the pile. $a_n = 2a_{n-1} + 1$ and $a_1 = 1$.

(3) Example: How many sequences of length n are there consistsing of zeros and ones with no two consecutive ones? (Call such a sequence admissible.)

Answer: $b_n = b_{n-1} + b_{n-2}$ with $b_0 = 1, b_1 = 2$. We just need to consider the last digit and divide it into two cases. This sequence is a variation of so-called *Fibonacci sequence*: $1, 1, 2, 3, 5, 8, 13, 21, \dots$ which satisfies $F_n = F_{n-1} + F_{n-2}$

(4) A *permutation* of $[n]$ is a bijective function from the set $[n]$ to itself. A *derangement* is a permutation which leaves no point fixed. That is, if π is derangement, then $\pi(i) \neq i$ for any $i \in [n]$. How many derangements are there?

Note that every permutation can be expressed as disjoint cycles with the smallest number as the element in each cycle. For example (124)(35) is the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$.

Answer: let $d(n)$ be the number of derangement of $[n]$. Then $d(n) = (n-1)d(n-1) + (n-1)d(n-2))$ for $n \geq 2$ (depending on whether n is in a 2-cycle or not).

(5) *Catalan number* We want to calculate a product $x_1 \cdot x_2 \cdots x_n x_{n+1}$. If we can only multiply two factors at a time, we have to put in brackets to make the expression well-defined. How many ways can we bracket such a product?

To form a bracketing, we have $n-1$ operations, and every number is in exactly one binary operation. We further assume that whenever possible, we will carry out the operations from left to right.

Suppose that a_1 is in the binary operation after $k-1$ numbers with $2 \leq k \leq n$. Then before the operation, we had $k-2$ operations (and $k-1$ elements) which have C_{k-1} ways to form it. After the operation, we have $n-k+1$ elements (remember we have a new one obtained from the operation of a_1), and need $n-k$ operations to go, which have C_{n-k+1} ways to form. So we have $C_{n+1} = \sum_{k=2}^{n} C_{k-1} C_{n-k+1}$, with $C_1 = C_2 = 1$. The number C_n is the n-th Catalan number.

Problem (k^5): Ways of connecting $2n-2$ points labelled $1, 2, \ldots, 2n2$ lying on a horizontal line by nonintersecting arcs above the line such that the left endpoint of each arc is odd and the right endpoint is even.