Math 432 Lec 05 Solve recurrence relations by characteristic equation method

Let a_n count an object formed from n elements. Like in induction, we sometimes can build a relation $a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-k})$ between a_n and $a_{n-1}, \ldots, a_{n-k}$. This is called a recurrence relation.

(1) *homogeneous linear recurrence relations*

For $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + f(n)$, the associated *characteristic polynomial* is $\phi(x) = x^k - c_1 x^{k-1} - \ldots - c_k x^0$, the *characteristic equation* is $\phi(x) = 0$, and its solutions are the *characteristic roots*.

Theorem. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ with multiplicities $d_1, d_2, \ldots, d_r$ be the distinct characteristic roots of a homogeneous linear recurrence relation (that is $f(n) = 0$) of order k with constant coefficients. The solutions have the form $a_n = \sum_i P_i(n)\alpha_i^n$, where each P_i is a polynomial of degree less than d_i. This solution is called *the general solution* to the recurrence.

For example, if the characteristic equation is $(x-1)(x-2)^3(x+3)^2 = 0$. Then the general solution would be $f_1 \cdot 1^n + (f_2 + f_3 n + f_4 n^2) \cdot 2^n + (f_5 + f_6 n)(-3)^n$, where $f_i, 1 \leq i \leq 6$ are constants to be determined by the initial terms.

Ex: Solve $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ with $a_0 = a_1 = 1$.

(2) *inhomogeneous linear recurrence relations*

We can follow three steps:

- Find the general solution b_n of the homogeneous relations;
- Find a particular solution d_n of the nonhomogeneous relation;
- Combine the general solution and the particular solution, and determine the constants arising in the general solutions such that the combined solution satisfies the initial conditions.

Here is a general rule to find a particular solution:

Let $f(n) = F(n)c^n$, where $F(n)$ is a polynomial of degree d. If c has multiplicity r as a characteristic root of the homogeneous part (r may be zero), then the recurrence has a solution of the form $P(n)n^r c^n$, where P is a polynomial of degree at most d.