Math 432 Lec 06 Solve recurrence using generating functions

Review of the homework and show that the Catalan number $C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$. The idea of the proof is show the bad ballot paths correspond bijectively to the paths from $(0,0)$ to $(n-1, n+1)$ (show injections in each direction).

Now we look at another way to solve (linear) recurrence (with constant coefficients):

(1) Definition of Generating functions:

With a sequence of numbers $a_1, a_2, \ldots, a_n, \ldots$, we can express these numbers in a *formal power series* $\sum_{n\geq 0} a_n x^n$, and this is called the generating function for a_n. Here "formal" indicates that x^n serves not as a number but rather as a placeholder forthe coefficient a_n. It is not really a function, and we don't care whether it is convergent or not. But if it is convergent, we will have a simple function for this power series.

Ex1: $a_k = \binom{n}{k}$, and we have $\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$.

Ex2: $a_n = 1$ and we have $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. Here a_n is the number of n-multisets formed from a single element (thus only one way).

(2) Extended Binomial Theorem: for $\alpha \in \mathcal{R}$ and $|x| < 1$,

$$(1+x)^\alpha = \sum_{k\geq 0} \binom{\alpha}{k} x^k, \text{ where } \binom{\alpha}{k} = \frac{(\alpha)(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{k!}$$

Proof from calculus (MacLaurin series): let $f(x) = \sum_{k\geq 0} a_k x^k$, and take n-th derivative of that. Now take $\alpha = -1, -n,$, we get some very useful generating functions:

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n, \qquad \frac{1}{(1-x)^k} = \sum_{n\geq 0} \binom{k+n-1}{k-1} x^n$$

(3) Solve recurrence relations using generating functions:

Theorem: Let $\alpha_1, \ldots, \alpha_r$ be distinct numbers satisfying the following equation for complex numbers $c_1, \ldots, c_k$ with $x_k \neq 0$

$$Q(x) = 1 - c_1 x - c_2 x^2 - \ldots - c_k x^k = \prod_{i=1}^{r} (1 - \alpha_i x)^{d_i}.$$

Then the following are equivalent for a sequence $\langle a \rangle$.

(a) $\langle a \rangle$ satisfies the recurrence $a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k}$ for $n \geq k$.

(b) $\langle a \rangle$ has generating function $P(x)/Q(x)$ for some polynomial P of degree less than k.

(c) $\langle a \rangle$ has generating function $\sum_{i=1}^{r} F_i(x)(1-\alpha_i x)^{-d_i}$, where each F_i is a polynomial of degree less than d_i.

(d) a_n for $n \geq 0$ is given by the formula $a_n = \sum_{i=1}^{r} P_i(n)\alpha_i^n$, where each P_i is a polynomial of degree less than d_i.

Ex: $a_n = a_{n-1} + n + 2^n$ with $a_1 = 1$.

$$a_n = a_{n-1} + n + 2^n, \quad n \geq 2, \quad a_1 = 1$$

For $n \geq 2$, $\quad a_n x^n = a_{n-1} x^n + n x^n + 2^n x^n$

so $\quad \sum_{n\geq 2} a_n x^n = \sum_{n \geq 2} a_{n-1} x^n + \sum_{n\geq 2} n x^n + \sum_{n \geq 2} (2x)^n$

Let $f(x) = \sum_{n \geq 1} a_n x^n$. Then we have

$$f(x) - a_1 x = x f(x) + x \left(\sum_{n\geq 2} x^n \right)' + \left(\frac{1}{1-2x} - 1 - 2x \right)$$

$$\Rightarrow (1-x) f(x) = x + x \left(\frac{1}{1-x} - 1 - x \right)' + \frac{1}{1-2x} - 1 - 2x$$

$$\Rightarrow f(x) = \cancel{\frac{x}{1-x}} + \frac{x}{(1-x)^3} - \cancel{\frac{x}{1-x}} + \frac{1}{(1-2x)(1-x)} - \frac{1+2x}{1-x}$$

$$= \frac{x}{(1-x)^3} + \left(\frac{2}{1-2x} - \frac{1}{1-x} \right) - \frac{1+2x}{1-x}$$

$$= x \sum_{n \geq 0} \binom{n+2}{2} x^n + 2 \sum_{n\geq 0} 2^n x^n - 2 \sum_{n \geq 0} x^n - 2x \sum_{n \geq 0} x^n$$

$$= \sum_{n \geq 1} \left(\binom{n+1}{2} + 2^{n+1} - 4 \right) x^n$$

So $\quad [x^n] f(x) = \binom{n+1}{2} + 2^{n+1} - 4, \quad n \geq 1$