**Math 432 Lec 08 Generating functions -2**

We may operate on the OGFs and on the expansions without losing equality. This may allow us to use OGF to prove something pleasantly.

Some basic propositions:

(1) *shifting the index.*

$$\sum_{k\geq 0}\binom{k}{r}x^k=\sum_{k\geq r}\binom{k}{r}x^k=\sum_{m\geq 0}\binom{m+r}{r}x^{m+r}=\frac{x^r}{(1-x)^{r+1}}.$$

Let $A, B, C$ be are the OGFs for $< a >, < b >, < c >$, respectively, then

(2) $c_n = a_n + b_n$ for all $n$ if and only if $C(x) = A(x) + B(x)$.

(3) $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ for all $n$ iff $C(x) = A(x)B(x)$.

(4) $b_n = \begin{cases} a_{n-k}, & \text{for } n \geq k \\ 0, & \text{for } n < k \end{cases}$ if and only if $B(x) = x^k A(x)$.

(5) $c_n = \sum_{i=0}^{n} a_i$ for all $i$ if and only if $C(x) = \frac{A(x)}{1-x}$.

(6) $b_n = \begin{cases} a_n \text{ for } n \text{ even} \\ 0 \text{ for } n \text{ odd} \end{cases}$ if and only if $B(x) = 0.5(A(x) + A(-x))$.

(7) $b_n = \begin{cases} 0, & \text{for } n \text{ even} \\ a_n, & \text{for } n \text{ odd} \end{cases}$ if and only if $B(x) = 0.5(A(x) - A(-x))$.

(8) $b_n = \begin{cases} a_k, & \text{for } n = mk \\ 0, & \text{otherwise} \end{cases}$ if and only if $B(x) = A(x^m)$.

(9) $b_n = na_n$ if and only if $B(x) = xA'(x)$.

Examples:

- $\sum_{i\geq 0}\binom{n}{2i}$.
  The sum is the value at $x = 1$ of $B(x) = \sum\binom{n}{2i}x^{2i}$. Let $A(x) = \sum_i\binom{n}{i}x^i = (1+x)^n$. Then $B(x) = \frac{A(x)+A(-x)}{2}$. So the sum is $B(1) = 2^{n-1}$.
- $\sum_{k\geq 0} k^m$.
  Consider $\sum_{k\geq 0} k^m x^k$. (work out for $m = 2$)
- $\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$.
  The convolution of $a_k = \binom{m}{k}$ and $b_k = \binom{n}{k}$. So the sum is $[x^r](1+x)^m(1+x)^n = [x^r](1+x)^{m+n} = \binom{m+n}{r}$.

- $\sum_{k=0}^{n} k(n-k)$.
  The convolution of $a_k = b_k = k$. So the sum is
$$[x^n](\sum_k kx^k)^2 = [x^n]\frac{x^2}{(1-x)^4} = [x^{n-2}]\frac{1}{(1-x)^4} = \binom{n+1}{3}$$

**Snake Oil method:** When $n$ is a parameter in a sum, we can always multiply by $x^n$ to form an OGF $A(x)$. Then interchange the order of summation in the expression for $A(x)$ and perform the new inner sum on $n$ explicitly.

Ex. $\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{m}{k}\binom{n}{k}2^k$.

Multiply both sides by $x^n$ and sum over $n$, and interchange the order of the summation.

$$LHS = \sum_k \binom{m}{k} x^{-k} \sum_{n\geq 0} \binom{n+k}{m} x^{n+k} = \frac{x^m}{(1-x)^{m+1}} \sum_k \binom{m}{k} x^{-k} = \frac{x^m(1+x^{-1})^m}{(1-x)^{m+1}} = \frac{(1+x)^m}{(1-x)^{m+1}}.$$

$$RHS = \sum_k \binom{m}{k} 2^k \sum_{n\geq 0} \binom{n}{k} x^n = \frac{1}{1-x} \sum_k \binom{m}{k} 2^k (\frac{x}{1-x})^k = \frac{1}{1-x}(1+\frac{2x}{1-x})^m = \frac{(1+x)^m}{(1-x)^{m+1}}.$$