Lec22 Ramsey Theory

Example: Among any six people, there are three mutual acquaintances or three mutual strangers.

Roughly speaking, Ramsey Theory states that for any k-coloring of the t-subsets of an n-set S, if n is large enough, then there is N-subset, whose elements have the same color. More precisely,

Definition 1. *Given $k, t, p_1, p_2, \ldots, p_k$, there exists an integer N, such that for any k-coloring of the t-subsets of an N-set S, there is a p_i-subset whose elements are colored with i^{th}-color. The smallest integer N is called the* **Ramsey Number** $R(p_1, p_2, \ldots, p_k; t)$.

Thus

(1) For $t = 1$, $R(p_1, p_2, \ldots, p_k; 1) = p_1 + p_2 + \ldots + p_k - (k-1)$, which is the the **strong form of pigeonhole principle**. That is, if we put $p_1 + p_2 + \ldots + p_k - (k-1)$ objects into k classes, then i^{th}-class has more than p_i objects for some i.

(2) For $t = 2$, that is the edge-coloring of an n-vertex complete graph. It states that if $n \geq R(p_1, p_2, \ldots, p_k)$, then for any k-edge-coloring of K_n, there is a complete subgraph K_{p_i} whose edges are all colored with i^{th}-color.

Especially, $R(3,3) = 6R(3,4) = 9$ (To show $R(3,4) \leq 9$, we prove that some vertex is incident with at most 2 red edges, thus at most 6 blue edges, and the 6 endpoints contain a monochromatic triangle)

We may also prove that $R(p,q) \leq R(p-1,q) + R(p,q-1)$. (how?)

(3) For $t \geq 3$, we color the edges of hypergraphs. For example, the meaning of $R(m,m;3)$ is that for any 2-coloring of the 3-subsets of an $R(m,m;3)$-set, there is an m-subset whose 3-subsets have the same color.

It is very hard to determine the exact values of Ramsey numbers. We only know very few of them.

- $R(n,2) = R(2,n) = n$.
- $R(3,n)$ for $n \leq 9$ are know. Especially $R(3,3) = 6$ and $R(3,4) = 9$.
- $R(p,q) \leq \binom{p+q-2}{p-1}$.
- $R(k,k)$ is of special interest to people. From above, we know $R(k,k) \leq 4^k$.
- For a lower bound of $R(k,k)$, Erdos used a probability method showing that if $\binom{n}{k}(1/2)^{\binom{k}{2}} < 1$, then $R(k,k) > n$. This shows that $R(k,k) > \sqrt{2}^k$. Note that $\binom{n}{k} < (ne/k)^k$.

Examples:

(1) *Happy End Problem* For an integer m, there is an integer $N(m)$ such that every set of at least $N(m)$ points in the plane (no three on a line) contains an m-subset forming a convex m-gon.

Proof. Fact one: Among any five points in the plane, four determine a convex quadrilateral. (why?)

Fact two: If very 4-subset of m-points in the plane form a convex quadrilateral, then the m points form a convex m-gon. (why?)

Now take $N = R(m, 5; 4)$ and color each 4-set red if it forms a convex gon, otherwise, color it blue. Then we will have an m-set so that every 4-subset forms a convex 4-gon. So those m points form a convex m-gon.

(2) *The Schur Theorem* Given $k > 0$, there exists an integer s_k such that every k-coloring of the integers $1, 2, \ldots, s_k$ yields monochromatic x, y, z (not necessarily distinct) satisfying $x + y = z$.

Proof. Let $s_k = R_k(3; 2)+1$. Let f be a k-coloring of the integers $1, 2, \ldots, s_k$. Let f' be a k-coloring of the 2-subsets (edges) of the set $\{1, 2, \ldots, s_k\}$ defined by $f'(\{a, b\}) = f(|a - b|)$.

Then by definition, there are three integers a, b and c (assume $a < b < c$) such that $c - b, b - a, c - a$ are of the same color. Let $x = c - b, y = b - a, z = c - a$, then $f(x) = f(y) = f(z)$ and $x + y = z$.

Remark: This theorem is a special case of the Van der Waerden Theorem which states that for any given positive integers l, k, there exists an integer $w(l.k)$ such that every k-coloring of $1, 2, \ldots, w(l, k)$ contains a monochromatic l-term arithmetic progression.

Defintion: Graph Ramsey Number $R(G, H)$ be the minimum n such that in every 2-edge-coloring of K_n, there exists either a monochromatic G or a monochromatic H.

Thm: $R(2k_2, 2K_2) = 5$.

Thm (Burr-Erdos-Spencer) if $m \geq 2$, then $R(mK_3, mK_3) = 5m$.

Proof: Let red graph be $K_{3m-1} + K_{1,2m-1}$. So $5m$ is a lower bound.

For upper bound, use induction on m. $m = 2$ is homework. For $m \geq 3$, we have $5m - 3m \geq R(3, 3) = 6$, so we can delete m monochromatic disjoint triangles. If they all have the same number, we are done. Otherwise, there is a red S and a blue triangle T. Among the 9 edges between S and T, we may assume there are at least 5 red ones. So we can find a red and a blue triangle on five vertices. Now use induction on the remaining $5m - 5$ vertices.