2. If $\mathbf{x}$ and $\mathbf{y}$ are production vectors, then the total cost vector associated with the combined production $\mathbf{x}+\mathbf{y}$ is precisely the sum of the cost vectors $T(\mathbf{x})$ and $T(\mathbf{y})$. ■

PRACTICE PROBLEMS

1. Suppose $T : \mathbb{R}^5 \to \mathbb{R}^2$ and $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A and for each $\mathbf{x}$ in $\mathbb{R}^5$. How many rows and columns does A have?
2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Give a geometric description of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.
3. The line segment from $\mathbf{0}$ to a vector $\mathbf{u}$ is the set of points of the form $t\mathbf{u}$, where $0 \le t \le 1$. Show that a linear transformation T maps this segment into the segment between $\mathbf{0}$ and $T(\mathbf{u})$.

1.8 EXERCISES

1. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find the images under T of $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

2. Let $A = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Find $T(\mathbf{u})$ and $T(\mathbf{v})$.

In Exercises 3–6, with T defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique.

3. $A = \begin{bmatrix} 1 & 0 & -3 \\ -3 & 1 & 6 \\ 2 & -2 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 2 & -5 & 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -6 \\ -4 \\ -5 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -8 & 8 \\ 0 & 1 & 2 \\ 1 & 0 & 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 6 \\ 3 \\ 10 \end{bmatrix}$

7. Let A be a 6×5 matrix. What must a and b be in order to define $T : \mathbb{R}^a \to \mathbb{R}^b$ by $T(\mathbf{x}) = A\mathbf{x}$?

8. How many rows and columns must a matrix A have in order to define a mapping from $\mathbb{R}^5$ into $\mathbb{R}^7$ by the rule $T(\mathbf{x}) = A\mathbf{x}$?

For Exercises 9 and 10, find all $\mathbf{x}$ in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ for the given matrix A.

9. $A = \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix}$

10. $A = \begin{bmatrix} 3 & 2 & 10 & -6 \\ 1 & 0 & 2 & -4 \\ 0 & 1 & 2 & 3 \\ 1 & 4 & 10 & 8 \end{bmatrix}$

11. Let $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and let A be the matrix in Exercise 9. Is $\mathbf{b}$ in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

12. Let $\mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$, and let A be the matrix in Exercise 10. Is $\mathbf{b}$ in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

In Exercises 13–16, use a rectangular coordinate system to plot $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, and their images under the given transformation T. (Make a separate and reasonably large sketch for each exercise.) Describe geometrically what T does to each vector $\mathbf{x}$ in $\mathbb{R}^2$.

13. $T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

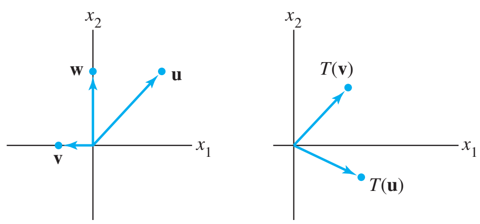
14. $T(\mathbf{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

15. $T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

16. $T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

17. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ into $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and maps $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Use the fact that T is linear to find the images under T of $2\mathbf{u}$, $3\mathbf{v}$, and $2\mathbf{u} + 3\mathbf{v}$.

18. The figure shows vectors **u**, **v**, and **w**, along with the images $T(\mathbf{u})$ and $T(\mathbf{v})$ under the action of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. Copy this figure carefully, and draw the image $T(\mathbf{w})$ as accurately as possible. [*Hint:* First, write **w** as a linear combination of **u** and **v**.]



19. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and maps $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

20. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps **x** into $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$. Find a matrix A such that $T(\mathbf{x})$ is $A\mathbf{x}$ for each **x**.

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. A linear transformation is a special type of function.

b. If A is a 3×5 matrix and T is a transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the domain of T is $\mathbb{R}^3$.

c. If A is an $m \times n$ matrix, then the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^m$.

d. Every linear transformation is a matrix transformation.

e. A transformation T is linear if and only if

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$$

for all $\mathbf{v}_1$ and $\mathbf{v}_2$ in the domain of T and for all scalars c_1 and c_2.

22. a. The range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of all linear combinations of the columns of A.

b. Every matrix transformation is a linear transformation.

c. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and if **c** is in $\mathbb{R}^m$, then a uniqueness question is "Is **c** in the range of T?"

d. A linear transformation preserves the operations of vector addition and scalar multiplication.

e. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ always maps the origin of $\mathbb{R}^n$ to the origin of $\mathbb{R}^m$.

23. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = mx + b$.

a. Show that f is a linear transformation when $b = 0$.

b. Find a property of a linear transformation that is violated when $b \neq 0$.

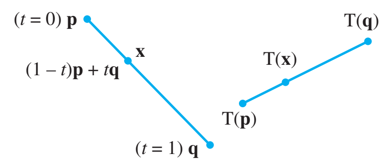
c. Why is f called a linear function?

24. An *affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, with A an $m \times n$ matrix and **b** in $\mathbb{R}^m$. Show that T is *not* a linear transformation when $\mathbf{b} \neq \mathbf{0}$. (Affine transformations are important in computer graphics.)

25. Given $\mathbf{v} \neq \mathbf{0}$ and **p** in $\mathbb{R}^n$, the line through **p** in the direction of **v** has the parametric equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$. Show that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ maps this line onto another line or onto a single point (a *degenerate line*).

26. a. Show that the line through vectors **p** and **q** in $\mathbb{R}^n$ may be written in the parametric form $\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$. (Refer to the figure with Exercises 21 and 22 in Section 1.5.)

b. The line segment from **p** to **q** is the set of points of the form $(1 - t)\mathbf{p} + t\mathbf{q}$ for $0 \leq t \leq 1$ (as shown in the figure below). Show that a linear transformation T maps this line segment onto a line segment or onto a single point.



27. Let **u** and **v** be linearly independent vectors in $\mathbb{R}^3$, and let P be the plane through **u**, **v**, and **0**. The parametric equation of P is $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ (with s, t in $\mathbb{R}$). Show that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ maps P onto a plane through **0**, or onto a line through **0**, or onto just the origin in $\mathbb{R}^3$. What must be true about $T(\mathbf{u})$ and $T(\mathbf{v})$ in order for the image of the plane P to be a plane?

28. Let **u** and **v** be vectors in $\mathbb{R}^n$. It can be shown that the set P of all points in the parallelogram determined by **u** and **v** has the form $a\mathbf{u} + b\mathbf{v}$, for $0 \leq a \leq 1, 0 \leq b \leq 1$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Explain why the image of a point in P under the transformation T lies in the parallelogram determined by $T(\mathbf{u})$ and $T(\mathbf{v})$.

29. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects each point through the x_2-axis. Make two sketches similar to Fig. 6 that illustrate properties (i) and (ii) of a linear transformation.

30. Suppose vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, \ldots, p$. Show that T is the zero transformation. That is, show that if **x** is any vector in $\mathbb{R}^n$, then $T(\mathbf{x}) = \mathbf{0}$.

31. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

In Exercises 32–36, column vectors are written as rows, such as $\mathbf{x} = (x_1, x_2)$, and $T(\mathbf{x})$ is written as $T(x_1, x_2)$.

32. Show that the transformation T defined by $T(x_1, x_2) = (x_1 - 2|x_2|, x_1 - 4x_2)$ is not linear.

33. Show that the transformation T defined by $T(x_1, x_2) = (x_1 - 2x_2, x_1 - 3, 2x_1 - 5x_2)$ is not linear.

34. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation that reflects each vector $\mathbf{x} = (x_1, x_2, x_3)$ through the plane $x_3 = 0$ onto $T(\mathbf{x}) = (x_1, x_2, -x_3)$. Show that T is a linear transformation. [See Example 4 for ideas.]

35. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation that projects each vector $\mathbf{x} = (x_1, x_2, x_3)$ onto the plane $x_2 = 0$, so $T(\mathbf{x}) = (x_1, 0, x_3)$. Show that T is a linear transformation.

36. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set, but $\{T(\mathbf{u}), T(\mathbf{v})\}$ is a linearly dependent set. Show that $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution. [*Hint:* Use the fact that $c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$ for some weights c_1 and c_2, not both zero.]

[**M**] In Exercises 37 and 38, the given matrix determines a linear transformation T. Find all $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$.

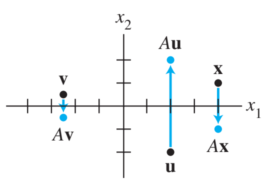
37. $\begin{bmatrix} 2 & 3 & 5 & -5 \\ -7 & 7 & 0 & 0 \\ -3 & 4 & 1 & 3 \\ -9 & 3 & -6 & -4 \end{bmatrix}$ **38.** $\begin{bmatrix} 3 & 4 & -7 & 0 \\ 5 & -8 & 7 & 4 \\ 6 & -8 & 6 & 4 \\ 9 & -7 & -2 & 0 \end{bmatrix}$

39. [**M**] Let $\mathbf{b} = \begin{bmatrix} 8 \\ 7 \\ 5 \\ -3 \end{bmatrix}$ and let A be the matrix in Exercise 37. Is $\mathbf{b}$ in the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$? If so, find an $\mathbf{x}$ whose image under the transformation is $\mathbf{b}$.

40. [**M**] Let $\mathbf{b} = \begin{bmatrix} 4 \\ -4 \\ -4 \\ -7 \end{bmatrix}$ and let A be the matrix in Exercise 38. Is $\mathbf{b}$ in the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$? If so, find an $\mathbf{x}$ whose image under the transformation is $\mathbf{b}$.

SG Mastering: Linear Transformations 1–34

SOLUTIONS TO PRACTICE PROBLEMS

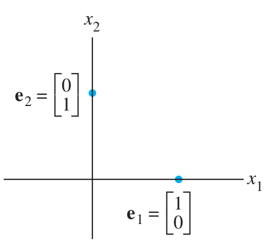


The transformation $\mathbf{x} \mapsto A\mathbf{x}$.

1. A must have five columns for $A\mathbf{x}$ to be defined. A must have two rows for the codomain of T to be $\mathbb{R}^2$.
2. Plot some random points (vectors) on graph paper to see what happens. A point such as $(4, 1)$ maps into $(4, -1)$. The transformation $\mathbf{x} \mapsto A\mathbf{x}$ reflects points through the x-axis (or x_1-axis).
3. Let $\mathbf{x} = t\mathbf{u}$ for some t such that $0 \le t \le 1$. Since T is linear, $T(t\mathbf{u}) = t\,T(\mathbf{u})$, which is a point on the line segment between $\mathbf{0}$ and $T(\mathbf{u})$.

1.9 THE MATRIX OF A LINEAR TRANSFORMATION

Whenever a linear transformation T arises geometrically or is described in words, we usually want a "formula" for $T(\mathbf{x})$. The discussion that follows shows that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is actually a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ and that important properties of T are intimately related to familiar properties of A. The key to finding A is to observe that T is completely determined by what it does to the columns of the $n \times n$ identity matrix I_n.



EXAMPLE 1 The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

With no additional information, find a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$.