Math 214 – Foundations of Mathematics
Homework 10

Due Nov 16, 2012

Your name

Each problem is worth 4 points.

1. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix}$ be permutations in S_5.
   Determine $\alpha \circ \beta$, $\beta \circ \alpha$, and β^{-1}.

2. Let $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$ be bijections. Show that $h : A \times B \to \mathbb{N}^2$ with $h(a, b) = (f(a), g(b))$ is also a bijection. (Remark: this fact was used to show that $A \times B$ is denumerable if A and B are.)

3. Show that if A and B are denumerable sets, then $A \cup B$ is also denumerable (hint: consider the cases when A and B are disjoint or not).

4. Prove that $S = \{(a, b) : a, b \in \mathbb{N}, a \geq 2b\}$ is denumerable.

5. For $k \in \mathbb{N}$, let $S_k = \{A \subset \mathbb{N} : |A| = k\}$. Show that $|S_2| = |\mathbb{N}|$ (hint: construct a bijection from S_2 to a subset of $\mathbb{N}^2$).

6. show that $|\mathbb{Q}| = |\mathbb{Q} - \{2\}|$.

7. (*Bonus, 4 points*) Using the definition of S_k from problem 5, show that
   (a) for all $k \in \mathbb{N}$, S_k is denumerable.
   (b) $\mathcal{S} = \bigcup_{k=1}^{\infty} S_k$ is denumerable.

8. (Bonus, 4 points) In class we mentioned that according to the "diagonal method", the elements in $\mathbb{N}^2$ can be listed as

$$(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), (1, 4), (2, 3), (3, 2), (4, 1), \ldots.$$

   Construct the explicit bijection from $\mathbb{N}^2$ to $\mathbb{N}$.